A bijection between $\mathbb{R}^2$ and $\mathbb{R}$

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July 14, 2011

When initially considering $\mathbb{R}$ (the real line) and $\mathbb{R}^2$ (the real plane), it may seem counterintuitive to say that both sets have the same size; or, in proper terminology, that $\left|\mathbb{R}^2\right| = |\mathbb{R}|$. However, thanks to a result in set theory known as the **Cantor–Bernstein–Schroeder theorem**, it's easy to show that this is indeed the case. In this article, I'll present a brief explanation of the theorem and then use it to prove the desired fact.

0.1 The Cantor–Bernstein–Schroeder theorem

The theorem states that for any two sets X and Y, if you can find injective functions $f : X \to Y$ and $g : Y \to X$, then there exists a bijective function $h : X \to Y$. From a set-theoretic perspective, this implies that $|X| = |Y|$.

To understand this theorem, one must first understand a few things about functions. A function is a mapping between two sets: its *domain* and its *codomain*. In other words, functions send values from the domain to the codomain. If a function f has a domain X and a codomain Y, then we denote it by $f : X \to Y$. Note that not every value in X and Y must be used. Unless otherwise mentioned, we will be discussing functions of the form $f : \mathbb{R} \to \mathbb{R}$; functions that map the real line to itself, and are thus visualized on the real plane.

A function is *injective* if every value in the codomain is mapped to by a value in the domain no more than once; in other words, if $f(a) = f(b)$, then $a = b$. An example of a function that is *not* injective would be $f(x) = x^2$, since $f(-1) = (-1)^2 = 1^2 = f(1)$. With careful thought, one may realize that if $f : X \to Y$ is injective then this implies that $|X| \leq |Y|$.

A function is *surjective* if every value in the codomain is mapped to by a value in the domain at least once. The previous example $f(x) = x^2$ would again be not surjective, since there is no value for x that produces a negative number. However, a function such as $f(x) = x^3$ would be. *Why?* Try to figure this one out on your own.

With these definitions in place, we can finally define what it means for a function to be bijective. A function is *bijective* if it is both *injective* and *surjective*. Ponder what this means for a moment. Combining the definitions of injectivity and surjectivity, we arrive at this: A function is bijective if each eleent in X corresponds to exactly one element in Y. Since $|X| \leq |Y|$ and

$|Y| \leq |X|$, we can conclude that bijectivity implies $|X| = |Y|$. It should start to become clear how a bijection may be useful. If we can define a bijection between two sets, then we have shown that **both sets have the same size**.

This is where Cantor–Bernstein–Schroeder comes into play. Since you'll find that in general it is difficult to define a bijection between arbitrary sets, this theorem allows us to only find a pair of injections, which is much easier, to prove that the two sets are equal in size. The details of how this theorem works are beyond the scope of this article, however. However, interested readers may undoubtedly find useful explanations through clever Google searches.

0.2 A bijection

To some, it may not be immediately evident that $|\mathbb{R}^2| = |\mathbb{R}|$. After all, the real plane seems to contain many more points than a single line. In fact, it contains infinitely many lines! However, we can now utilize Cantor–Bernstein–Schroeder to more easily verify the truth of this claim.

Firstly, we shall find an injection $f : \mathbb{R} \to \mathbb{R}^2$. This is easier than one may think. The function $f(x) = (x, 0)$, where (x, y) is a member of $\mathbb{R}^2$, is an injective function. A proof for the injectivity of this function is simple enough to be omitted.

However, finding such a suitable function $g : \mathbb{R}^2 \to \mathbb{R}$ is slightly more difficult. Through some educated guessing, the function $g(x, y) = x + xy$ was chosen. Since it is not obvious that g is injective, a proof shall be given. Note that in order to prove that g is injective, we must show that $f(a, b) = f(c, d)$ implies $(a, b) = (c, d)$. However, this yields an unwieldy proof, so an alternative definition shall be used. Instead, we shall say that g is injective if $g(a, b) = g(a + c, b + d)$ implies $c, d = 0$. One may easily realize that these two definitions are equivalent, since c and d are arbitrary. The proof is as follows:

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\begin{aligned}
g(a, b) &= g(a + c, b + d) \\
a + ab &= (a + c) + (a + c) \cdot (b + d) \\
a + ab &= (a + c) \cdot (1 + (b + d)) \\
a + ab &= a \cdot (1 + b + d) + c \cdot (1 + b + d) \\
a + ab &= a + ab + ad + c \cdot (1 + b + d) \\
0 &= ad + c \cdot (1 + b + d)
\end{aligned}
$$

After solving for d, substituting it in and evaluating the result, it is clear that c and d must both be zero. Thus g is injective. Therefore, since we have found injective functions $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$, by Cantor–Bernstein–Schroeder there must exist a bijection between $\mathbb{R}$ and $\mathbb{R}^2$–and this implies that they are the same size. Done.