On the notion of a finite set

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Mr. W. Sierpinski has given in his book *The axiom of Zermelo M. and its role in set theory and analysis* a new definition of finite set. This definition differs essentially from the fact that it does not depend on either the notion of a natural number or a function, both of which are used to denote correspondence in the usual sense. The definition in question is as follows:

> "Consider a class K of sets, each of which satisfies the following conditions:
>
> 1. Any set containing a single element belongs to the class K,
> 2. If A and B are two sets belonging to the class K, their union $A \cup B$ also belongs to K.
>
> Let us call *finite* every set which belongs to a class K that satisfies the conditions 1 and 2."

As it is known, the set of all objects (if it exists) has paradoxical properties. In contrast to a well-known theorem by G. Cantor, the cardinality of this set under Sierpinski's definition would not be strictly lesser than that of the cardinality of the class of all its subsets. The same is true of the class composed of all the sets containing a single element; therefore, the classes K do not uphold Cantor's theorem. Taking this into account, one could question the very existence of any such class K.

By changing the definition of Mr. Sierpinski so as to eliminate this disadvantage, I get the following definition; a set M is finite when the class of all its nonempty subsets is the unique class that satisfies the conditions:

1. Its elements are nonempty subsets of M;

2. Any set containing a single element of M belongs to this class;

3. If A and B belong to this class, then their union $A \cup B$ also belongs to this class.

We show that this definition of a finite set is equivalent to the ordinary one. In other words, if a set is finite by the proposed definition, it is sufficient that the number of elements can be expressed by a natural number. The concept of a natural number is assumed to be known.

Indeed, let M be a set whose number of elements can be expressed by a natural number and Z any class satisfying the conditions 1 to 3. We will show that any subset of M belongs to Z. Under condition 2, Z must contain all subsets of M with a single element. Then, for any subset contained in Z of n elements, we may form a new subset of $n+1$ elements by condition 3. It follows by induction that Z contains all subsets of M. Since Z is identical to all subsets of M, it is unique. Thus, any set whose number of elements can be expressed by a natural number is a finite set in our sense.

On the other hand, suppose that the number of elements of a given set M cannot be expressed by a natural number. Let Z denote the class of all subsets of M whose number of elements can be expressed by a natural number. This class obviously satisfies the conditions 1 to 3. Yet M does not belong to Z, and consequently Z is not identical to the class of all subsets of M. Therefore, the class of all subsets of M is not the only class satisfying conditions 1 to 3, and M is not finite in our sense. $\square$